Time-Uniform Confidence Sequences from Universal Gambling

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Joint work with Alankrita Bhatt (Caltech)

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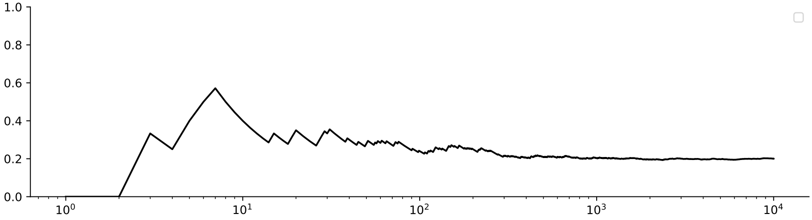

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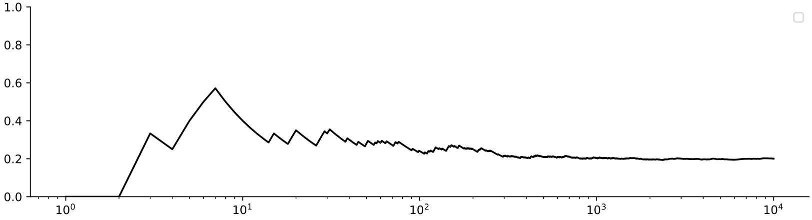

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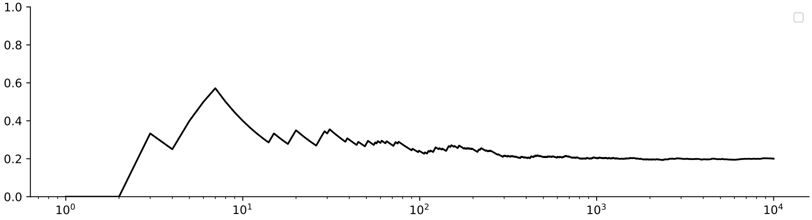

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- For reliable inference, we need to quantify confidence

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which is equivalent to

$$\mu \in \left(\hat{\mu}_t - \sqrt{\frac{1}{t}\frac{\mathrm{Var}(Y)}{\delta}}, \hat{\mu}_t + \sqrt{\frac{1}{t}\frac{\mathrm{Var}(Y)}{\delta}}\right) \text{ with prob. } \ge 1 - \delta$$

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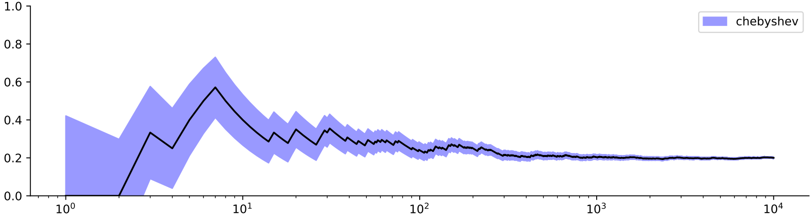

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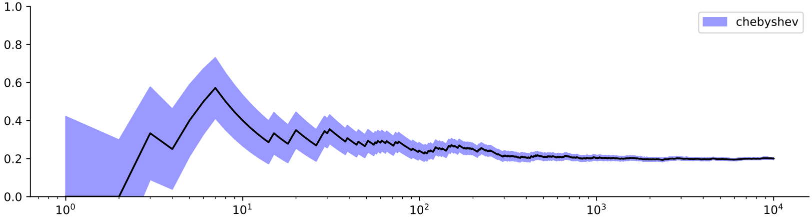

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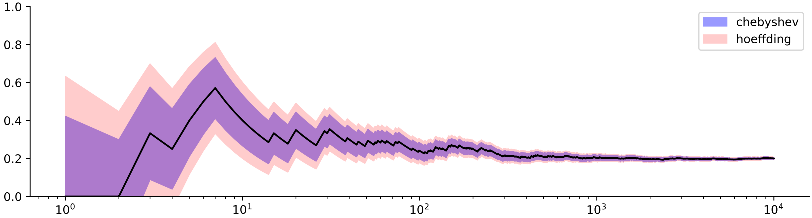

Confidence Sets

- A set $C_t(\delta)$ is a confidence set for μ at level $1-\delta$, if $C_t(\delta)$ is a function of Y^t and
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- For such online data processing, we need to construct a sequence of confidence intervals that is valid at any time

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- Originally studied by Darling and Robbins (1967); Lai (1976), and recently resurrected by some statisticians (Ramdas et al., 2020; Waudby-Smith and Ramdas, 2020a,b; Howard et al., 2021) and computer scientists (Jun and Orabona, 2019; Orabona and Jun, 2021)

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Ville's inequality (Ville, 1939)

For a nonnegative supermartingale sequence $(W_t)_{t=0}^{\infty}$ with $W_0 > 0$,

$$\mathsf{P}\left\{\sup_{t\geq 1} \frac{W_t}{W_0} \geq \frac{1}{\delta}\right\} \leq \delta$$

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- As a slight detour, let's review canonical gambling problems and some universal gambling strategies

Universal Gambling

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- The recursive equation:

$$\mathsf{W}_t = \mathsf{W}_{t-1} 2 q_t^{\mathbb{1}\{y_t=1\}} (1 - q_t)^{\mathbb{1}\{y_t=0\}} = \mathsf{W}_{t-1} 2 q(y_t | y^{t-1})$$

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- Cumulative wealth: starting with $\$W_0$,

$$W_T = W_0 \prod_{t=1}^{T} 2 q(y_t | y^{t-1}) = W_0 2^T q(y^T),$$

where $q(y^T) := \prod_{t=1}^{T} q(y_t | y^{t-1})$

Universality and Minimax Optimality

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- The best strategy is called minimax optimal

$$\min_{q} \max_{p \in \mathcal{P}} \max_{y^T} \log \frac{\mathrm{W}^p(y^T)}{\mathrm{W}^q(y^T)}$$

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- Note: $\dfrac{\mathsf{W}^p(y^T)}{\mathsf{W}^q(y^T)} = \dfrac{\mathsf{W}_0 2^T p(y^T)}{\mathsf{W}_0 2^T q(y^T)} = \dfrac{p(y^T)}{q(y^T)}$ by definition

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$\therefore$ universal compression $\rightarrow$ universal betting!

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- Krichevsky–Trofimov (KT) probability assignment (Krichevsky and Trofimov, 1981)

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$$q_{\mathsf{KT}}(1|y^{t-1}) := \frac{1}{t}\left(\sum_{i=1}^{t-1} y_i + \frac{1}{2}\right)$$

- Asymptotically minimax optimal (Xie and Barron, 2000)

$$\max_{\theta\in[0,1]} \max_{y^T} \log \frac{p_\theta(y^T)}{q_{\mathsf{KT}}(y^T)} = \frac{1}{2}\log T + \frac{1}{2}\log\frac{\pi}{2} + o(1)$$

Mixture Probability

- The KT probability $q_{\mathsf{KT}}(\cdot|y^{t-1})$ is induced by a mixture probability, i.e.,

$$q_{\mathsf{KT}}(y^T) \equiv \int_0^1 p_\theta(y^T)\,\mathrm{d}\pi(\theta)$$

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- So, mixture is nice!

Horse Race

- Horses: $1, 2, \ldots, m$

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- KT strategy: $q_{\mathrm{KT}}(y^T) := \int_{\Delta_{m-1}} p_{\boldsymbol{\theta}}(y^T) \, \mathrm{d}\pi(\boldsymbol{\theta})$, where $\pi(\boldsymbol{\theta}) = \mathrm{Dir}(\boldsymbol{\theta}|\frac{1}{2}, \ldots, \frac{1}{2})$

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Stock Investment

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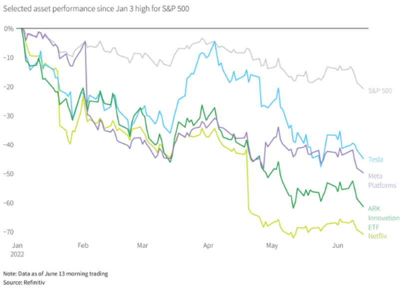

Image credit: https://www.reuters.com/article/usa-stocks-bearmarket-idCAKCN2N61PI

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$$\mathbf{x}_t = (x_{t1}, \ldots, x_{tm}) \in \mathcal{M} \subseteq \mathbb{R}^m_{\geq 0},$$

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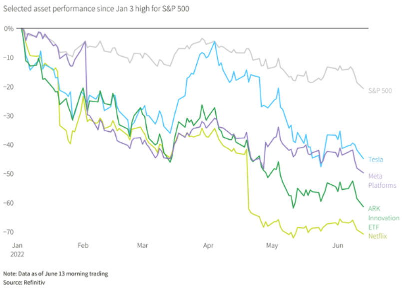

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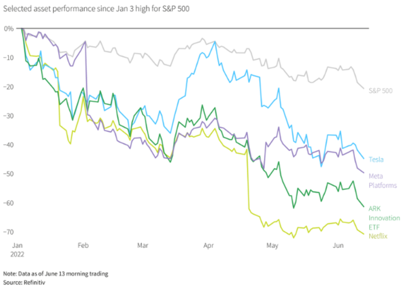

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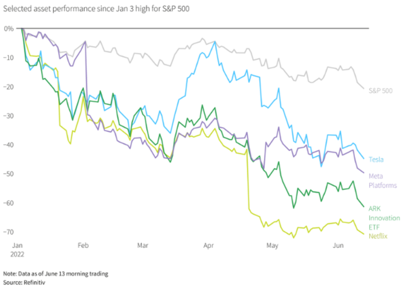

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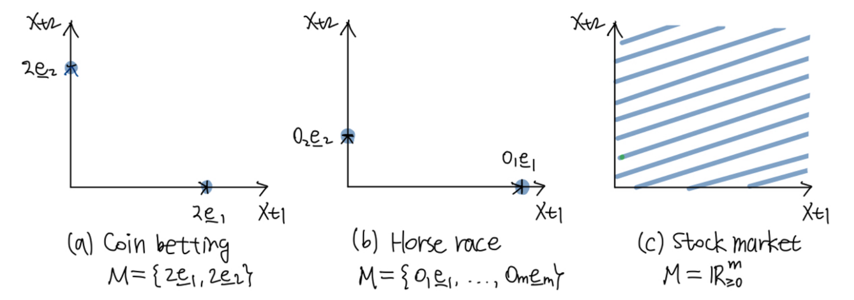
Special Cases



From Probability Assignment to Portfolio Selection

- By distributive law,

$$\mathrm{W}(\mathbf{x}^T) = \mathrm{W}_0 \prod_{t=1}^{T} \langle \mathbf{b}(\mathbf{x}^{t-1}), \mathbf{x}_t \rangle = \mathrm{W}_0 \sum_{y^T \in [m]^T} \Big(\prod_{t=1}^{T} b(y_t \,|\, \mathbf{x}^{t-1}) \Big) \mathbf{x}^T(y^T),$$

where $\mathbf{x}^T(y^T) := x_{1y_1} \dots x_{Ty_T}$ = (multiplicative gain of the extremal portfolio y^T)

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- A probability induced portfolio: for a probability $q(y^T)$, define

$$\mathrm{W}^q(\mathbf{x}^T) := \mathrm{W}_0 \sum_{y^T \in [m]^T} q(y^T) \mathbf{x}^T(y^T),$$

which is achieved by a causal bettor $\mathbf{b}^q$ defined to satisfy

$$\mathrm{W}^q(\mathbf{x}^t) = \mathrm{W}^q(\mathbf{x}^{t-1}) \langle \mathbf{b}^q(\mathbf{x}^{t-1}), \mathbf{x}_t \rangle$$

Portfolio Selection $\equiv$ Probability Assignment

Theorem

$$\sup_{p\in\mathcal{P}} \sup_{\mathbf{x}^T} \frac{\mathsf{W}^p(\mathbf{x}^T)}{\mathsf{W}^q(\mathbf{x}^T)} = \sup_{p\in\mathcal{P}} \sup_{y^T} \frac{p(y^T)}{q(y^T)}$$

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Proof

$$\sup_{\mathbf{x}^n} \sup_{p\in\mathcal{P}} \frac{W^p(\mathbf{x}^n)}{W^q(\mathbf{x}^n)} \geq \sup_{y^n\in[m]^n} \sup_{p\in\mathcal{P}} \frac{W^p(\mathbf{e}_{y_1}\ldots\mathbf{e}_{y_n})}{W^q(\mathbf{e}_{y_1}\ldots\mathbf{e}_{y_n})} = \sup_{y^n\in[m]^n} \sup_{p\in\mathcal{P}} \frac{p(y^n)}{q(y^n)}$$

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Lemma $\star$ (Cover, 2006, Lemma 16.7.1)

For $a_i, b_i \geq 0$, we have $\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \leq \max_{j\in[n]} \frac{a_j}{b_j}$, where $\frac{0}{0} := 0$

Example: Constant Rebalanced Portfolios

- $\mathcal{P}_{\text{i.i.d.}} = \{\text{i.i.d. categorical probabilities}\} = \{p_{\boldsymbol{\theta}}(\cdot) \colon \boldsymbol{\theta} \in \Delta_{m-1}\}$

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- Note: for horse race, UP is equivalent to the simple KT strategy

From Universal Gambling to Confidence Sequences

Supermartingales from Gambling

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Proof.

For every t, $\mathsf{E}[\mathsf{W}_t|\mathbf{x}^{t-1}] = \mathsf{W}_{t-1}\langle \mathbf{b}_t, \mathsf{E}[\mathbf{x}_t|\mathbf{x}^{t-1}]\rangle \le \mathsf{W}_{t-1}\langle \mathbf{b}_t, \mathbb{1}\rangle = \mathsf{W}_{t-1}$

Examples

- Coin betting: $\mathbf{x}_t = (2Y_t, 2(1 - Y_t))$, $Y_t \in \{0, 1\}$
  - fair if $\mathsf{E}[Y_t | Y^{t-1}] = \frac{1}{2}$ (e.g., $Y_t \sim$ i.i.d. $\mathrm{Bern}(\frac{1}{2})$)

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- Two-horse race: $\mathbf{x}_t = (o_1 Y_t, o_2(1-Y_t))$, $Y_t \in \{0,1\}$
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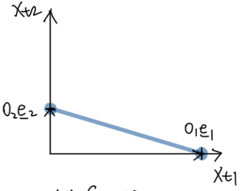

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- If $m = \mu$, any wealth process from $\text{CTHR}(m)$ is martingale
- If $m \neq \mu$, there exists a causal betting strategy whose wealth process from $\text{CTHR}(m)$ is **strictly** submartingale

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- For CTHR(m), we play a strategy $(\mathbf{b}(Y^{t-1}; m))_{t=1}^{\infty}$ and get $(\mathrm{W}(Y^t; m))_{t=1}^{\infty}$

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- If we collect all m whose corresponding wealth never exceeds W_0/δ by then, it forms a time-uniform confidence set with level $1-\delta$

Confidence Sequence from CTHR(m)

- Formally, if we define

$$C_t(Y^t;\delta) := \left\{ m \in (0,1) \colon \sup_{1\le i\le t} \frac{\mathsf{W}(\mathbf{x}^i;m)}{\mathsf{W}_0} < \frac{1}{\delta} \right\},$$

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- Orabona and Jun (2021) empirically showed that applying Cover's UP gives tight confidence sequences

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$(C_t^{\mathsf{KT}}(Y^t;\delta))_{t=1}^{\infty}$ is a time-uniform confidence interval with level $1-\delta$

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- **Note**: the size of the interval behaves as $\sqrt{\frac{2}{t}\log\frac{1}{\delta} + \frac{1}{t}\log t + o(1)}$ for $t \gg 1$, which is comparable to $\sqrt{\frac{2}{t}\log\frac{1}{\delta}}$ from the standard Hoeffding[1]

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- An alternative approach (Ryu and Bhatt, 2022)
  - Recall that Cover's UP is defined as a mixture of wealths of CRPs
  - Consider a tight lower bound of the CRP wealth and take a mixture over the lower bounds

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For any $n \in \mathbb{N}$ and $m \in (0,1)$, we have

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- Lower-bound the logarithm by moments of y, i.e., $(1, y, \ldots, y^{2n})$

Key Lemma for the Proof

Lemma (Generalization of (Fan et al., 2015, Lemma 4.1))

For an integer $\ell \geq 1$, if we define

$$f_\ell(t) := \begin{cases} \left(\log(1+t) - \sum_{k=1}^{\ell-1} (-1)^{k+1} \frac{t^k}{k}\right) \Big/ \left((-1)^\ell \frac{t^\ell}{\ell}\right) & \text{if } t > -1 \text{ and } t \neq 0, \\ -1 & \text{if } t = 0, \end{cases}$$

then $t \mapsto f_\ell(t)$ is continuous and strictly increasing over $(-1, \infty)$

- Note: Fan et al. (2015) considered $\ell = 2$, i.e.,

$$f_2(t) = \begin{cases} \frac{\log(1+t) - t}{t^2/2} & \text{if } t > -1 \text{ and } t \neq 0, \\ -1 & \text{if } t = 0 \end{cases}$$

A Lower Bound on the Cumulative Wealth of CRP

- Since it is easy to check $\phi_n(x;\boldsymbol{\rho},\eta)\phi_n(x;\boldsymbol{\rho}',\eta') = \phi_n(x;\boldsymbol{\rho}+\boldsymbol{\rho},\eta+\eta')$,

Lemma

For any $n \in \mathbb{N}$, $m \in (0,1)$, $b \in [0,1]$, and $y^t \in [0,1]^t$, we have

$$\log \frac{\mathrm{W}_t^b(y^t;m)}{\mathrm{W}_0} \ge \log \phi_n\left(\frac{\bar{b}}{\bar{m}}; \boldsymbol{\rho}_n(y^t;m), \eta_n(y^t;m)\right)$$

if $m < b < 1$, where $\eta_n(y^t;m) := \sum_{i=1}^t (1-\frac{y_i}{m})^{2n}$ and

$$(\boldsymbol{\rho}_n(y^t;m))_k := \sum_{i=1}^t \left\{\left(1-\frac{y_i}{m}\right)^{2n} - \left(1-\frac{y_i}{m}\right)^k\right\} \quad \text{for } k = 1,\ldots,2n-1$$

- Lower-bound the logarithm by moments of y^t, i.e., $(\sum_{i=1}^t y_i^j)_{j=1}^{2n}$
- Complexity from $O(t)$ to $O(n)$

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- We can construct a time-uniform confidence interval using this "mixture of wealth lower bounds"!
- We call this LBUP(n), where n is the approximation order

Caveats

- Computational bottleneck: computing the normalization constant $Z_n(\boldsymbol{\rho}, \eta)$ of the form

$$\int_0^1 x^\eta \exp\Big(\sum_{k=0}^{2n-1} a_k x^k\Big)\, \mathrm{d}x$$

  - Hence, $O(1)$ per round in principle, but may take longer than running exact UP due to numerical integration steps
- Larger n leads to better approximation, but with increased numerical instability; $n = 2$ or $n = 3$ empirically work well
- Bad approximation in a small sample regime
  - Hybrid UP: run UP for the first few samples and switch to LBUP

Evolution of Wealth Processes

- The horizontal lines indicate an example threshold $\ln \frac{1}{\delta} \approx 2.996$ for $\delta = 0.05$

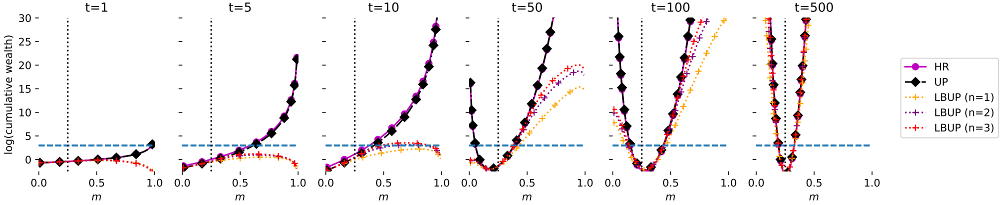

Figure: An i.i.d. Bern(0.25) process

Evolution of Wealth Processes

- The horizontal lines indicate an example threshold $\ln \frac{1}{\delta} \approx 2.996$ for $\delta = 0.05$

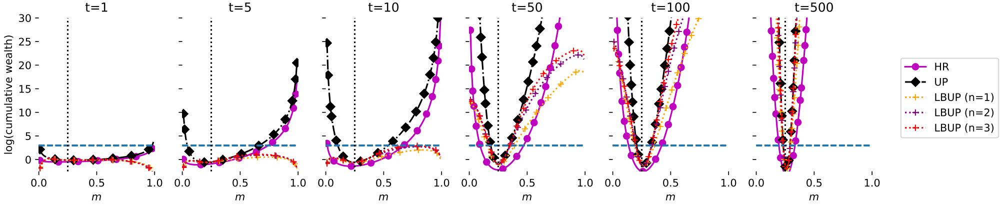

Figure: An i.i.d. Beta(1,3) process

Evolution of Wealth Processes

- The horizontal lines indicate an example threshold $\ln \frac{1}{\delta} \approx 2.996$ for $\delta = 0.05$

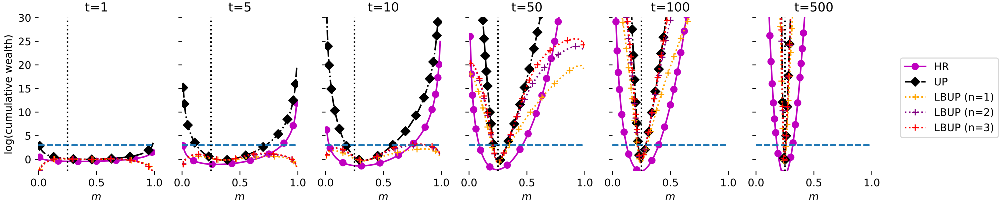

Figure: An i.i.d. Beta(10,30) process

Experiments

- Confidence sequences with level 0.95 (i.e., $\delta = 0.05$)
- CB: betting strategy from another gambling construction
- HR: KT strategy
- UP: exact Cover's UP strategy
- LBUP: proposed lower-bound approach
- HybridUP: run exact UP for the first few steps and switch to LBUP
- PRECiSE (Orabona and Jun, 2021)

Experiments

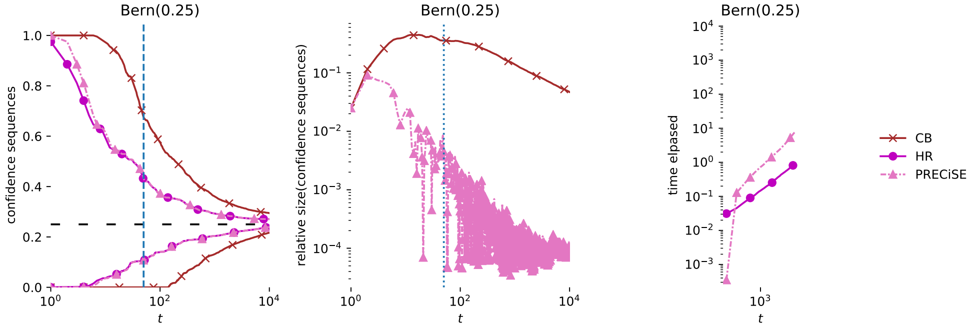

Figure: With i.i.d. Bern(0.25) processes

Experiments

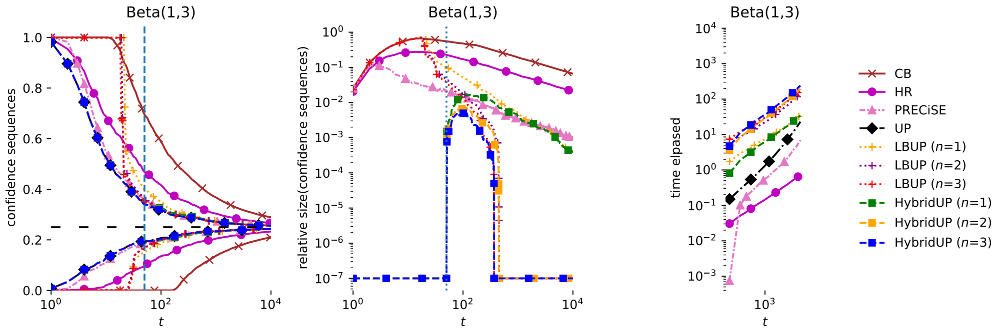

Figure: With i.i.d. Beta(1,3) processes

Experiments

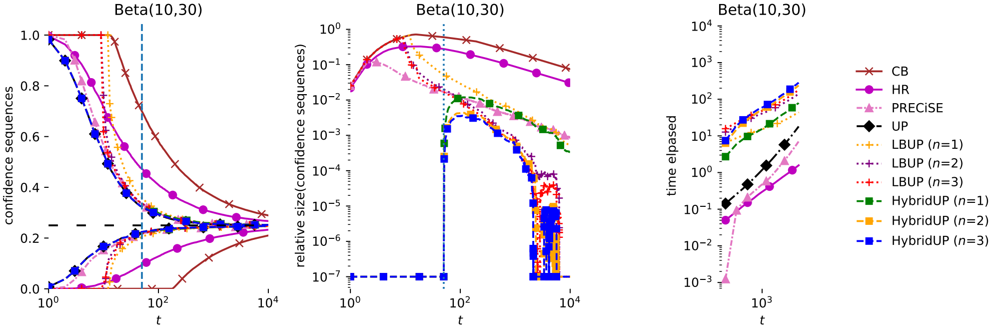

Figure: With i.i.d. Beta(10,30) processes

Take-Home Messages

- Confidence sequence is an important tool in modern data science
- Gambling with respect to probability induced strategies $\equiv$ probability assignment
- Confidence sequences from universal portfolios are very tight with small samples, but suffers high complexity
- They can be "efficiently" approximated by a mixture of lower bounds approach!

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- They can be "efficiently" approximated by a mixture of lower bounds approach!

Q. Can we construct a time-uniform confidence set for bounded vectors? Yes!

Q. Can there be a gambling other than CTHR(m) that corresponds to some other statistics applications?

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